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Eigenvalue spacing statistics of a four-matrix model of some four-by-four random matrices

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Abstract
A simple four-matrix model consisting of some 4×4 real and imaginary random matrices and three parameters is introduced. It is shown that the eigenvalue spacing statistics of an ensemble of such matrices can be used to describe transitions between all the Wigner surmises of random matrix theory. Formulae for the nearest-neighbour spacing distributions of the ensemble are given for various parameter values.

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The nearest-neighbour spacing distribution (NNSD) is one of the most basic means of studying the statistics of a set of eigenvalues or levels. It is often calculated before any other statistical measure in order to help classify the system under study. This can be achieved by comparing to known results from random matrix theory (RMT), where formulae for the NNSDs of eigenvalues from the various Gaussian ensembles [i.e. the Gaussian orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensembles] are known [1]. Usually, however, the NNSD of a set of levels is compared to the Wigner surmises, which are exact for Gaussian ensembles of 2×2 random matrices only. This is done because the surmises are in fact superb analytical approximations for the NNSDs of eigenvalues from Gaussian ensembles of large $N \times N$ random matrices [1–3] and because they are easily programmed. The Wigner surmises for the NNSDs of eigenvalues from the GOE ($\beta = 1$), GUE ($\beta = 2$) and GSE ($\beta = 4$) are [4]

$$P_W(S;\beta) = \mathcal{A}(\beta) S^{\beta} \exp(-\mathcal{B}(\beta) S^2), \tag{1a}$$

where

$$\mathcal{A}(\beta) = 2\frac{\left[\Gamma\left(\frac{\beta+2}{2}\right)\right]^{\beta+1}}{\left[\Gamma\left(\frac{\beta+1}{2}\right)\right]^{\beta+2}} \qquad \text{and} \qquad \mathcal{B}(\beta) = \frac{\left[\Gamma\left(\frac{\beta+2}{2}\right)\right]^{2}}{\left[\Gamma\left(\frac{\beta+1}{2}\right)\right]^{2}}. \tag{1b}$$

S is the normalized spacing and β is known as the 'level repulsion' parameter—for small values of S the behaviour $P_W(S;\beta) \sim S^{\beta}$ results.

Many times one finds that the NNSD of a set of levels cannot be described by usual forms such as the Wigner surmises or the Poisson distribution [$P(S) = \exp(-S)$] for randomly occurring levels. Therefore, it is useful to have intermediate distribution functions with which one can compare. Intermediate (or transitional) distributions come in several varieties and although many examples are given below we do not intend this to be a complete survey of the subject. One of the most well-known is the Brody distribution [5, 6], which is used to describe transitions from Poisson statistics to Wigner–GOE statistics, although other descriptions for the Poisson-to-GOE intermediate region are possible [2, 7–13]. The more exotic semi-Poisson-to-Wigner–Ginibre transition has recently been identified [14] (but it has not yet been used to describe eigenvalue statistics) as have the GUE-to-Ginibre [15] and the GUE-to-Poisson-like [16] transitions. The NNSD for the generalized Wigner surmise [17], for 2×2 random real symmetric matrices, provides another means of describing level statistics that do not follow the standard forms, and analytical forms for NNSDs of different Wigner–GOE-to-Wigner–GUE transitions [10, 18, 19] are also known. Transitions in terms of other types of statistics can also be studied but are not of immediate interest for the present study—these include level densities for a random matrix model that incorporates both GUE and Poisson statistics [20], number variance for non-Hermitian matrices that interpolate between the GUE, the Ginibre and the Poisson ensembles [21], the Dyson–Mehta statistic for Poisson-to-GOE transitions [11, 12], and n-level cluster functions for GOE-to-GUE and GSE-to-GUE transitional systems, see [1, 3] for a summary.

The purpose of our work is to introduce a simple random matrix model that can be used to transition through all the Wigner surmises, including the Wigner surmise for the Ginibre ensemble [22] which results from setting $\beta = 3$ in equation (1). Our model, represented by H below, is a four-matrix model built from some special 4×4 random matrices and three parameters (α_1, α_2 and α_3) which are allowed to take on values ranging from 0 to 1:

$$H = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ c & 0 & b & 0 \\ 0 & c & 0 & b \end{pmatrix} + \mathrm{i}\alpha_1 \begin{pmatrix} 0 & 0 & d & 0 \\ 0 & 0 & 0 & -d \\ -d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 & 0 & e \\ 0 & 0 & -e & 0 \\ 0 & -e & 0 & 0 \\ e & 0 & 0 & 0 \end{pmatrix} + \mathrm{i}\alpha_3 \begin{pmatrix} 0 & 0 & 0 & f \\ 0 & 0 & f & 0 \\ 0 & -f & 0 & 0 \\ -f & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Note that a, b, c, d, e and f are all real and independent—they are zero-centred Gaussian distributed (the variances of a and b are twice that of the rest of the variables). For convenience we will set the variances of the a and b distributions to two and the variances of the c, d, e and f distributions to one. If we let $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $x = c + \mathrm{i}d$ and $y = e + \mathrm{i}f$ then

$$H = \begin{pmatrix} a & 0 & x & y \\ 0 & a & -y^* & x^* \\ x^* & -y & b & 0 \\ y^* & x & 0 & b \end{pmatrix}, \tag{3}$$

which is the $N = 2$ form of the GSE, in terms of complex matrix entries (see for example [3]). The structure of H is such that all eigenvalues are real and doubly degenerate for all allowed values of the α-parameters. The degenerate eigenvalues are easily found to be

$$E_\pm = \tfrac{1}{2}\left\{(a+b) \pm \left[(a-b)^2 + 4c^2 + 4\alpha_1^2 d^2 + 4\alpha_2^2 e^2 + 4\alpha_3^2 f^2\right]^{1/2}\right\}, \tag{4}$$

from which we get the spacing

$$\begin{aligned} s &= E_+ - E_- \\ &= \left[(a-b)^2 + 4c^2 + 4\alpha_1^2 d^2 + 4\alpha_2^2 e^2 + 4\alpha_3^2 f^2\right]^{1/2} \\ &= 2\left(g^2 + c^2 + \alpha_1^2 d^2 + \alpha_2^2 e^2 + \alpha_3^2 f^2\right)^{1/2}, \end{aligned} \tag{5}$$

Table 1. NNSDs of eigenvalues for various H ensembles having only binary α-parameters.

No	α_1	α_2	α_3	Wigner-	NNSD
1	0	0	0	GOE	$P_W(S;1)$
2	1	0	0	GUE	$P_W(S;2)$
3	0	1	0	GUE	$P_W(S;2)$
4	0	0	1	GUE	$P_W(S;2)$
5	1	1	0	Ginibre	$P_W(S;3)$
6	1	0	1	Ginibre	$P_W(S;3)$
7	0	1	1	Ginibre	$P_W(S;3)$
8	1	1	1	GSE	$P_W(S;4)$

where g is zero-centred Gaussian distributed with a variance of one. We now refer to [23] and note that studying the statistics of s, or more conventionally $S = s/\bar{s}$, is analogous to studying radial distances of points chosen (randomly) from a multidimensional Gaussian distribution (the term Gaussian point process used in [23] has the potential of being confused with pre-existing terminology [24] and so we will not use it here). Therefore, it is relatively easy to infer the spacing statistics of eigenvalues from an ensemble of H matrices for different α-parameters. For example, the NNSDs for H ensembles having only binary α-parameters are exactly given by the Wigner surmises, $P_W(S;\beta)$, as shown in table 1. It is interesting to note that, for systems 5, 6 and 7, cubic level repulsion is present for a random matrix ensemble that has all real eigenvalues, unlike Ginibre's general ensemble [22] which has complex eigenvalues.

We can now move on to studying transitional systems by allowing the α-parameters to continuously take on values between 0 and 1. A Wigner–GOE-to-Wigner–GUE transition is obtained by setting $0 \leqslant \alpha_1 \leqslant 1$ and $\alpha_2 = \alpha_3 = 0$. Aside from the matrix size and the degenerate eigenvalues this is essentially equivalent to the French–Kota–Pandey–Mehta two-matrix model of RMT [18, 25–27]. It is then obvious that the NNSD of eigenvalues from an ensemble of such matrices is [3, 18, 19, 28]

$$P_H(s;\alpha_1) = \frac{s}{4\left(1-\alpha_1^2\right)^{1/2}} \exp\left(\frac{-s^2}{8}\right) \operatorname{erf}\left[\left(\frac{1-\alpha_1^2}{8\alpha_1^2}\right)^{1/2} s\right], \tag{6a}$$

where

$$\bar{s} = \sqrt{\frac{8}{\pi}}\left[\left(1-\alpha_1^2\right)^{-1/2} \arctan\left\{\left(1-\alpha_1^2\right)^{1/2}/\alpha_1\right\} + \alpha_1\right]. \tag{6b}$$

This spacing distribution can also be found by comparing equation (5) to the radial distance of a point chosen from a three-dimensional Gaussian distribution, where two of the coordinates of the point are chosen from a zero-centred Gaussian having a variance of one and the third coordinate is chosen from a zero-centred Gaussian having a variance of α_1^2 (see [23] once again, and note the important factor of 2 when making this comparison).

A Wigner–GUE-to-Wigner–Ginibre transition is obtained by setting $\alpha_1 = 1, 0 \leqslant \alpha_2 \leqslant 1$ and $\alpha_3 = 0$. The NNSD can again be found by comparing equation (5) to the radial distance of a point chosen from a multidimensional Gaussian distribution; this time, however, we consider a point in four dimensions. Three of the coordinates of the point are chosen from a zero-centred Gaussian having a variance of one but the fourth coordinate is chosen from a zero-centred Gaussian having a variance of α_2^2. It can then be shown that

$$P_H(s;\alpha_2) = \frac{1}{16\pi\alpha_2} \int_0^{\pi} \exp\left[-\frac{s^2}{8}\left(\sin^2\xi + \frac{\cos^2\xi}{\alpha_2^2}\right)\right] s^3 \sin^2\xi \, \mathrm{d}\xi. \tag{7}$$

We have not been able to find a compact analytical solution to this problem (recall that we are also interested in solving for $\bar{s}$) and will therefore solve it numerically.

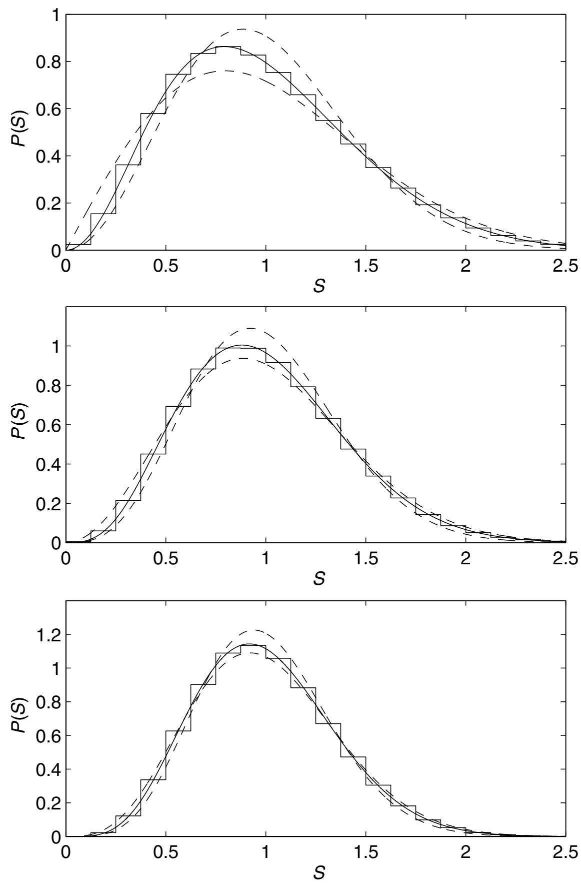

Figure 1. Numerical studies of various transitions (histograms). The top, middle and bottom panels show Wigner–GOE-to-Wigner–GUE, Wigner–GUE-to-Wigner–Ginibre and Wigner–Ginibre-to-Wigner–GSE intermediate systems, respectively. The α-parameter values used are given in table 2. Dashed lines represent the corresponding Wigner surmises $[P_W(S;\beta)]$ for each panel. Solid lines represent the appropriate forms of $P_H(S;\alpha_1,\alpha_2,\alpha_3)$.

Finally, we consider a Wigner–Ginibre-to-Wigner–GSE transition by setting $\alpha_1 = \alpha_2 = 1$ and $0 \leqslant \alpha_3 \leqslant 1$. This is analogous to studying the radial distance of a point chosen from a five-dimensional Gaussian distribution, where four of the coordinates of the point are chosen from a zero-centred Gaussian having a variance of one but the fifth coordinate is chosen from a zero-centred Gaussian having a variance of α_3^2. The NNSD is given by

$$P_H(s;\alpha_3) = \frac{1}{64\sqrt{2\pi}\alpha_3}\int_0^{\pi} \exp\left[-\frac{s^2}{8}\left(\sin^2\psi + \frac{\cos^2\psi}{\alpha_3^2}\right)\right] s^4 \sin^3\psi \, \mathrm{d}\psi. \tag{8}$$

Again, we settle for a numerical integration.

For completeness we performed numerical studies of various transitional systems. An ensemble of one-million 4×4 random matrices was formed, from which we could extract 10^6 spacings, for the three transitional systems given in table 2. Note that an unfolding procedure for the eigenvalues is not needed since we are dealing with only two (degenerate) eigenvalues for each matrix. Shown in figure 1 are the results of the numerical studies carried out. Note that $S = s/\bar{s}$ is a normalized spacing and that $\bar{s}$ was obtained by numerical integration for

Table 2. Transitional systems presently studied. See figure 1 for results.

No	α_1	α_2	α_3	Type of transition
1	0.4	0	0	Wigner–GOE-to-Wigner–GUE
2	1	0.4	0	Wigner–GUE-to-Wigner–Ginibre
3	1	1	0.4	Wigner–Ginibre-to-Wigner–GSE

systems 2 and 3. The appropriate forms of $P_H(S; \alpha_1, \alpha_2, \alpha_3)$ are also shown in the figure for comparison.

To summarize, we have introduced a four-matrix model of some special 4×4 random matrices that can be used to transition between the well-known Wigner surmises of RMT. A complete Wigner–GOE-to-Wigner–GUE-to-Wigner–Ginibre-to-Wigner–GSE transition is possible by continuously varying the three parameters of the model. Although we chose to vary one parameter (i.e. either α_1, α_2 or α_3) at a time, we are by no means restricted to doing so. That is, a more general form of equation (2) results when the α-parameters are all allowed to take on any value ranging from 0 to 1 simultaneously. In this way the NNSDs of other transitional systems can be studied using the same techniques as above. Furthermore, by generalizing the variances of the matrix element distributions in equation (2), as was done in [17], NNSDs of generalized Wigner surmises, beyond random real symmetric matrices, can also be studied.

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